Exam Calculus 2

9 April 2018, 9:00-12:00

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[6+6+3=15 Points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) For which unit vector $\boldsymbol{u} = v\mathbf{i}+w\mathbf{j}$ with $v^2+w^2 = 1$, does the directional derivative $D_{\boldsymbol{u}}f(0,0)$ exist?

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. **[7+8=15 Points.]** Let $u : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto u(x,y,z)$ be a C^2 function. By defining spherical coordinates according to $(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, the function $u(x,y,z)$ can be considered as a function $f(\rho,\theta,\phi)$.

   (a) Express $\frac{\partial f}{\partial \theta}$ in terms of partial derivatives with respect to x, y and z of the function u.

   (b) Conversely the function $f(\rho,\theta,\phi)$ can be considered as a function $u(x,y,z)$. Suppose that the function f depends only on ρ (i.e. f is independent of θ and ϕ). Show that in this case

$$\frac{\partial^2}{\partial x^2}u(x,y,z) + \frac{\partial^2}{\partial y^2}u(x,y,z) + \frac{\partial^2}{\partial z^2}u(x,y,z) = \frac{2}{\rho}f'(\rho) + f''(\rho) .$$

3. **[4+4+7=15 Points.]** Consider the helix parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k},$$

   where a and b are positive constants.

   (a) Determine the length of the helix and its parametrization by arclength s.

   (b) At each point on the helix, determine the unit tangent vector $\mathbf{T}$ and the curvature of the helix κ.

   (c) Let $\mathbf{N}$ be the unit vector with direction $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}$ and let $\mathbf{B}$ be the unit vector defined as $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Compute $\mathbf{B}$ and show that $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{B} = -\tau\mathbf{N}$ for some $\tau \in \mathbb{R}$. Determine τ.

— **please turn over** —

4. **[3+6+6=15 Points]** Let S be the unit sphere in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$.

   (a) Compute the tangent plane of S at the point $(x_0, y_0, z_0) = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$.

   (b) Use the Implicit Function Theorem to show that near the point $(x_0, y_0, z_0) = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, the sphere S can be considered to be the graph of a function f of x and y. Compute the partial derivatives of f with respect to x and y and show that the tangent plane found in (a) coincides with the graph of the linearization of f at $(x_0, y_0) = (1/\sqrt{3}, 1/\sqrt{3})$.

   (c) Use the method of Lagrange multipliers to determine the points on S where $g(x, y, z) = xy^2z^3$ has maxima and minima, respectively.

5. **[5+5+5=15 Points]** For constants $a, b \in \mathbb{R}$, define the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\mathbf{F}(x, y, z) = ax\sin(\pi y)\,\mathbf{i} + (x^2\cos(\pi y) + bye^{-z})\,\mathbf{j} + y^2e^{-z}\,\mathbf{k}.$$

   (a) Show that $\mathbf{F}$ to be conservative requires $a = 2/\pi$ and $b = -2$.

   (b) Determine a scalar potential for $\mathbf{F}$ for the values of a and b given in part (a).

   (c) For the values of a and b given in part (a), compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve parametrized by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j} + \sin(2t)\,\mathbf{k}$$

   with $t \in [0, \pi/2]$.

6. **[8+7=15 Points]** For $r > 0$, let S_r denote the sphere of radius r with center at the origin, oriented with outward normal. Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is of class C^1 and is such that

$$\oiint_{S_r} \mathbf{F} \cdot d\mathbf{S} = ar + b \qquad (1)$$

   for fixed constants a and b.

   (a) Compute

$$\iiint_D \nabla \cdot \mathbf{F}\, dV,$$

   where $D = \{(x, y, z) \in \mathbb{R}^3 \mid 25 \le x^2 + y^2 + z^2 \le 49\}$.

   (b) Suppose that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^2 and Equation (1) holds for any $r > 0$. What conditions does this place on the constants a and b?